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Notes and Corrections on "Asymptotic Constancy Criteria of Solution of Linear Parabolic Volterra Difference Equations"

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Abstract—In this paper, we make some notes and corrections on "Asymptotic Constancy Criteria of Solutions of Linear Parabolic Volterra Difference Equations" [1].

Keywords—Parabolic Volterra difference equations, Neutral type, Asymptotic constancy, Von Neumann boundary conditions, Positive and negative coefficients.

In the recent paper, Shi *et al.* [1] studied the asymptotic constancy of solutions of linear parabolic Volterra difference equations of neutral type of form

$$\Delta_2\left(u_{m,n}-\sum_{k=1}^{\infty}r_{k,n}u_{m,n-\alpha_k}\right)+\sum_{i=1}^{\infty}p_{i,n}u_{m,n-\beta_i} - \sum_{j=1}^{\infty}q_{j,n}u_{m,n-\gamma_j}=d\Delta_1^2u_{m-1,n+1}, \tag{1}$$

$$\text{for } m=1,\dots,M \quad\text{and}\quad n\in z^+(n_0)=n_0,n_0+1,\dots,$$

with the homogeneous Von Neumann boundary conditions (NBC)

$$\Delta_1 u_{o,n}=\Delta_1 u_{M,n}=0, \qquad \text{for } n\in z^+(n_0), \tag{2}$$

and initial conditions (IC)

$$u_{m,l}=\theta_{m,l}, \qquad \text{for } m=1,\dots,M \quad\text{and}\quad l\in\bar z(n_0)=\{\dots,n_o-1,n_0\}, \tag{3}$$

satisfies

$$\|\Theta\|=\sup\{|\Theta_{m,l}| \text{ for } m=1,\dots,M \text{ and } l\in\bar z(n_0)\}<\infty, \tag{4}$$

where Δ_1, Δ_1^2, and Δ_2 are forward partial difference operators such that $\Delta_1u_{m,n}=u_{m+1,n}-u_{m,n}$, $\Delta_1^2u_{m,n}=\Delta_1(\Delta_1u_{m,n})$, and $\Delta_2u_{m,n}=u_{m,n+1}-u_{m,n}$ for $m=1,\dots,M, n\in z=\{\dots,-1,0,1,\dots\}$. In the following, Shi *et al.* [1] assumed that $r_{k,n}\in z^+(1)\times z^+(n_0)\to R$, $p_{i,n}\in z^+(1)\times z^+(n_0)\to R^+$, $q_{j,n}\in z^+(1)\times z\to R^+$, $B_i\ge\gamma_i$ for $i\in z^+(1)$, $d\in R^+$ and that there exists a constant $c>0$ such that $h_{i,n}\ge cq_{i,n-\beta_i+\gamma_i}$ for $i\in z^+(1)$ and sufficiently large n, where $h_{i,n}=p_{i,n}-q_{i,n-\beta_i+\gamma_i}$.

By constructing Liapunov sequences, Shi *et al.* [1] obtained the following asymptotic constancy theorem for equation (1).

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THEOREM 1. *(See [1, Theorem 2.1].) Assume that*

$$\limsup_{n\to\infty}\left(\sum_{k=1}^{\infty}|r_{k,n+1}|+\sum_{j=1}^{\infty}\sum_{\mu=n+1-\beta_j+\gamma_j}^{n}q_{j,\mu}\right)=\delta<1, \tag{c$_1$}$$

$$\begin{aligned}\limsup_{n\to\infty}&\left[\delta\left(\frac{1+\sum_{i=1}^{\infty}h_{i,n+1+\alpha_i+\beta_i}}{\sum_{i=1}^{\infty}h_{i,n+1+\beta_i}}\right)+\sum_{i=1}^{\infty}\sum_{\lambda=n+1-\beta_i}^{n+1}h_{i,\lambda+\beta_i}\right.\\&\left.+\frac{\sum_{i=1}^{\infty}(q_{i,n+1+\gamma_i}+h_{i,n+1+\beta_i})\sum_{\lambda=n+2-\beta_i}^{n+1}h_{i,\lambda+2\beta_i}}{\sum_{i=1}^{\infty}h_{i,n+1-\beta_i}}\right]<2,\end{aligned} \tag{c$_2$}$$

and that

$$\begin{aligned}\limsup_{n\to\infty}&\left[\delta+\sum_{i=1}^{\infty}\sum_{\lambda=n+1-\beta_i}^{n}h_{i,\lambda+\beta_i}+\sum_{k=1}^{\infty}|r_{k,n+1+\alpha_k}|\right.\\&\left.+\sum_{i=1}^{\infty}\beta_i(q_{i,n+1+\gamma_i}+h_{i,n+1+\beta_i})\right]<2.\end{aligned} \tag{c$_3$}$$

Then every solution of IBVP (1)–(3) tends to a constant as $n\to\infty$.

One can point out in the proof of [1, Theorem 2.1], there are very serious errors such that the theorem does not hold. Now abstract a set of sentences in [1, pp. 74,75] as follows.

By Step II, we know

$$\sum_{m=1}^{M}\left(u_{m,n}-\sum_{k=1}^{\infty}r_{k,n}u_{m,n-\alpha_k}-\sum_{j=1}^{\infty}\sum_{\mu=n-\beta_j+\gamma_j}^{n-1}q_{j,\mu}u_{m,\mu-\gamma_j}\right)\to\sqrt{M\omega},\qquad\text{as } n\to\infty.$$

We will now prove that $\sum_{m=1}^{M}u_{m,n}\to\sqrt{M\omega}/(1-\delta)$ as $n\to\infty$. In the matter of fact, we let

$$\limsup_{n\to\infty}\sum_{m=1}^{M}u_{m,n}=\alpha\quad\text{and}\quad\liminf_{n\to\infty}\sum_{m=1}^{M}u_{m,n}=\beta.$$

Choose a subsequence $\{n_i\}$ of $\{n\}$ such that $\lim_{i\to\infty}\sum_{m=1}^{M}u_{m,n_i}=\alpha$. Let $\varepsilon>0$ be sufficiently small such that

$$\begin{aligned}\sum_{m=1}^{M}u_{m,n_i}&=\sum_{m=1}^{M}\left(u_{m,n_i}-\sum_{k=1}^{\infty}r_{k,n}u_{m,n_i-\alpha_k}-\sum_{j=1}^{\infty}\sum_{\mu=n_i-\beta_i+\gamma_i}^{n_i-1}q_{j,\mu}u_{m,\mu-\gamma_j}\right)\\&\quad+\sum_{m=1}^{M}\left(\sum_{k=1}^{\infty}r_{k,n}u_{m,n_i-\alpha_k}+\sum_{j=1}^{\infty}\sum_{\mu=n_i-\beta_i+\gamma_i}^{n_i-1}q_{j,\mu}u_{m,\mu-\gamma_j}\right) &&\text{(c$_4$)}\\&\le\sqrt{M\omega}\,(\alpha+\varepsilon)\,\delta, &&\text{(c$_5$)}\end{aligned}$$

and $\alpha\le\sqrt{M\omega}+(\alpha+\varepsilon)\delta$. Letting $\varepsilon\to0$, we obtain $\alpha\le\sqrt{M\omega}+\alpha\varepsilon$, i.e., $\alpha\le\sqrt{M\omega}/(1-\delta)$. Similarly, we can show that

$$\beta\ge\frac{\sqrt{M\omega}}{1-\delta}. \tag{c$_6$}$$

Hence, $\alpha=\beta=\sqrt{M\omega}/(1-\delta)$.

One can see that since $\{u_{m,n}\}$ is assumed to be any solution of IBVP (1)–(3), the sign of α cannot be determined. Thus, after

$$\sum_{m=1}^{M} u_{m,n_i-\alpha_k} \quad \text{and} \quad \sum_{\mu=n_i-\beta_i+\gamma_i}^{n_i-1} u_{m,\mu-\gamma_i},$$

having been magnified into $(\alpha+\varepsilon)$, when

$$\sum_{k=1}^{\infty} |r_{k,n}| + \sum_{j=1}^{\infty} \sum_{\mu=n_i-\beta_i+\gamma_i}^{n_i-1} q_{j,\mu} u_{m,\mu-\gamma_i}$$

magnified into δ, it cannot be determined that whether (c_4) is magnified or reduced. Hence, (c_4) cannot follow (c_5) unless (c_1) is changed into

$$\lim_{n\to\infty} \left(\sum_{k=1}^{\infty} |r_{k,n+1}| + \sum_{j=1}^{\infty} \sum_{\mu=n+1-\beta_j+\gamma_j}^{n-1} q_{j,\mu} \right) = \delta < 1. \tag{5}$$

Similarly, since the sign of β cannot be determined, it cannot be defined whether (c_4) is magnified or reduced. Further, since $r_{k,n} \in Z^+(1) \times z^+(n_0) \to R$, from (c_4), only get

$$\beta \geq \sqrt{M\omega} + \sum_{m=1}^{M} \left(\sum_{k=1}^{\infty} |r_{k,n}| u_{m,n_i-\alpha_k} - \sum_{j=1}^{\infty} \sum_{\mu=n_i-\beta_j+\gamma_j}^{n_i-1} q_{j,\mu} |u_{m,\mu-\gamma_j}| \right).$$

Thus, from (c_4), one cannot get (c_6). Hence, (c_4) cannot follow (c_6) unless (c_1) is changed into (c_5) and $r_{k,n} \in Z^+(1) \times Z^+(n_0) \to R$ is changed into $r_{k,n} \in z^+(1) \times z^+(n_1) \to R^+$.

The above mentioned errors can be avoided and the result presented in [1] remains a valid subject to the following corrections: (c_4) is changed into (c_5) and $r_{k,n} \in z^+(1) \times z^+(n_0) \to R$ is changed into $r_{k,n} \in z^+(1) \times z^+(n_1) \to R^+$.

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